

MAS115: HOMEWORK 3

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1. THE SQUARE-ROOT OF 2

Here, we're going to investigate a solution of the equation

$$x^2 = 2. \tag{1}$$

Definition 1.1. The positive solution to equation (1) is denoted $\sqrt{2}$.

Lemma 1.2. Any non-zero rational number can be written in the form $\frac{a}{b}$, where a and b are coprime integers.

Proof. Let q be any non-zero rational number. Then, by definition, $q = \frac{c}{d}$ for some integers c and d , possibly sharing factors. Let h denote the highest common factor of c and d , and write $a = c/h$ and $b = d/h$. Then a and b are coprime integers and $q = \frac{c}{d} = \frac{c}{d} \cdot \frac{1/h}{1/h} = \frac{a}{b}$, as required. \square

Theorem 1.3. The real number $\sqrt{2}$ is irrational.

Proof. Suppose not, and $\sqrt{2}$ is rational. Then, by Lemma 1.2, $\sqrt{2} = \frac{a}{b}$, where a and b share no common factors. Thus, squaring, $2b^2 = a^2$. It follows that a^2 is even, and hence so is a . Write $a = 2m$ for some $m \in \mathbb{Z}$. Then $2b^2 = (2m)^2 = 4m^2$, so $b^2 = 2m^2$. As before, this means that b is even. But then a and b share a factor of 2, which is a contradiction. Hence $\sqrt{2}$ is irrational. \square