MAS115: HOMEWORK 4

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1. Lagrange Interpolation

Lagrange interpolation is a method used to fit smooth polynomial curves to sets of points in the plane. Suppose we have n distinct points in the plane (where n is an integer greater than 1), with no two x-coordinates equal. Then there is a polynomial $f(x)$ known as the *Lagrange polynomial* of degree at most $n-1$ which passes through the points. In some sense the Lagrange polynomial is the simplest smooth curve that fits the points.

1.1. Quadratic case.

Quadratic case. Suppose we have three points as described above, (x_0, y_0), (x_1, y_1) and (x_2, y_2) say. Let

$$\begin{aligned} p_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 \\ p_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 \\ p_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2 \end{aligned}$$

and put $f(x) = p_0(x) + p_1(x) + p_2(x)$. Notice that

$$p_0(x_0) = \frac{(x_0-x_1)(x_0-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 = y_0,$$

whilst $p_0(x_1) = p_0(x_2) = 0$. Similarly $p_1(x_1) = y_1$ (with $p_1(x_0) = 0$ and $p_1(x_2) = 0$) and $p_2(x_2) = y_2$ (with $p_2(x_0) = 0$ and $p_2(x_1) = 0$). It follows that $f(x_0) = y_0$, $f(x_1) = y_1$ and $f(x_2) = y_2$. Further, each of p_0, p_1 and p_2 have degree two. Hence $f(x)$ is a polynomial of degree at most two[1] which passes through (x_0, y_0), (x_1, y_1) and (x_2, y_2); that is, $f(x)$ interpolates our three chosen points.

Here, $f(x)$ is the Lagrange polynomial for the chosen points.

Looking at the interpolation for the points $(0,0)$, $(\frac{\pi}{4}, \frac{1}{\sqrt{2}})$ and $(\frac{3\pi}{4}, \frac{1}{\sqrt{2}})$ which all lie on the curve $y = \sin x$, we end up with $f(x) = \frac{16}{3\pi^2\sqrt{2}}x(\pi - x)$.

[1]It is possible that $f(x)$ has degree one or zero. For the points $(0,0)$, $(1,1)$ and $(2,2)$ we find that $f(x) = x$ (degree 1), whereas for the points $(0,0)$, $(1,0)$ and $(2,0)$ we find that $f(x) = 0$ (degree 0).

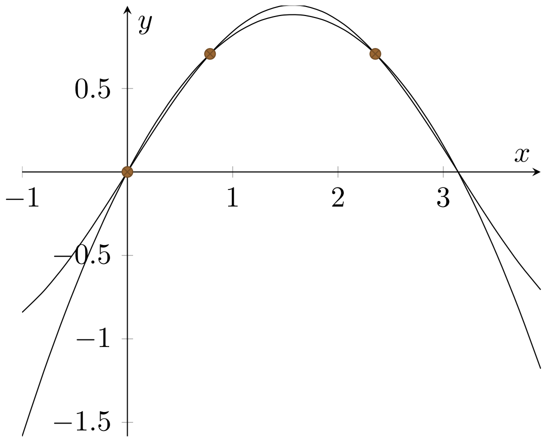

FIGURE 1. The graphs of $y = \frac{16}{3\pi^2\sqrt{2}}x(\pi - x)$ and $y = \sin x$

1.2. **General case.** It is easy to see how this theory generalises. Given n points, (x_i, y_i) for $0 \le i < n$, as described in the introduction, construct polynomials $p_i(x)$ for $0 \le i < n$ by

$$p_i(x) = \prod_{j \neq i} \frac{(x - x_j)}{(x_i - x_j)} y_i.$$

Then, putting $f(x) = \sum_{i=0}^{n-1} p_i(x)$, we have created an interpolating polynomial of degree at most $n - 1$. For details, see the Wikipedia page [1].

Figure 2 illustrates the case where $n = 4$, with a cubic curve interpolating the points $(-2, -4)$, $(-1, 1)$, $(0, 2)$ and $(1, 2)$. Here, the Lagrange polynomial turns out to be $f(x) = \frac{1}{2}(x^3 - x^2 + 4)$.

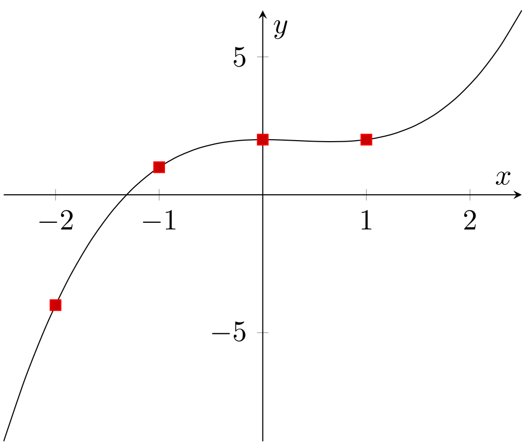

FIGURE 2. The graph of $y = \frac{1}{2}(x^3 - x^2 + 4)$, viewed as a Lagrange polynomial

References

[1] Wikipedia contributors, *Lagrange polynomial*, Wikipedia, The Free Encyclopedia. Visited 26 October 2012, updated 25 October 2012, `http://en.wikipedia.org/wiki/Lagrange_polynomial`.